Partial Derivatives

We will be interested in calculus-like problems for a function of several variables. For example:

- When does a function of several variables reach a maximum?
- How do we approximate a value of a function at a point near the point at which we know the value of the function?
- What is the shortest path to climb up the mountain?
- What is the velocity of the wind, given the atmospheric pressure function?

For this, a fundamental tool is the partial derivative.

<u>Definition</u> For a function $f(x,y)$ in two variables, the partial derivative of $f(x,y)$ with respect to x is denoted either $f_x(x,y)$ or $\frac{\partial f}{\partial x}(x,y)$ (or just $\frac{\partial f}{\partial x}$), and is obtained by just differentiating by x, treating y as a constant.

<u>Example</u> If $f(x,y) = xy + x^2 + y^2 - ye^y$,

$$\frac{\partial f}{\partial x} = y + 2x + \underbrace{0 - 0}_{\substack{\text{anything that doesn't} \\ \text{involve } x \text{ should have} \\ \frac{\partial f}{\partial x} \text{ go away!}}} = y + 2x$$

If $f(x,y) = x^y$, $\quad \dfrac{\partial f}{\partial x} = yx^{y-1}$.

If $f(x,y) = \dfrac{x}{y}$ $\quad \dfrac{\partial f}{\partial x} = \dfrac{1}{y}$.

Similarly, you could take the partial derivative with respect to y, denoted $f_y(x,y)$ or $\dfrac{\partial f}{\partial y}$.

<u>Example</u> If $f(x,y) = xy + x^2 + y^2 - ye^y$,

$$\frac{\partial f}{\partial y} = x + 0 + 2y - (e^y + ye^y) = x + 2y - e^y - ye^y.$$

If $f(x,y) = x^y$, $\quad \dfrac{\partial f}{\partial y} = x^y \ln x$.

If $f(x,y) = \dfrac{x}{y}$, $\quad \dfrac{\partial f}{\partial y} = -\dfrac{x}{y^2}$.

You can obviously do this for functions of 3 variables.

<u>Example</u> If $f(x,y,z) = \dfrac{x}{y-z}$, then

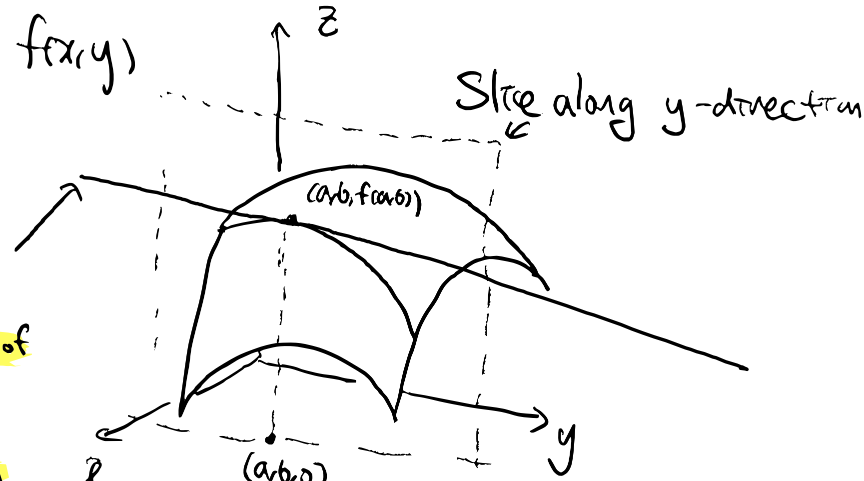
$$\frac{\partial f}{\partial x} = \frac{1}{y-z}$$

$$\frac{\partial f}{\partial y} = -\frac{x}{(y-z)^2}$$

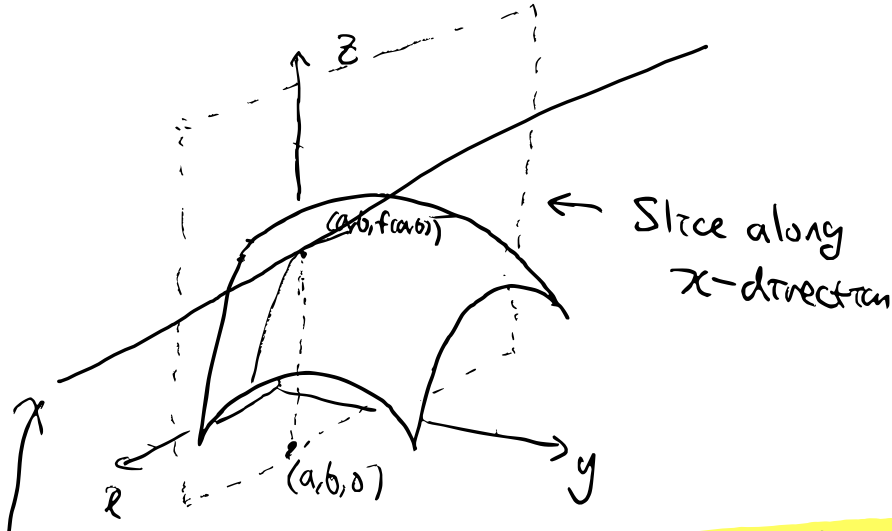
$$\frac{\partial f}{\partial z} = -\frac{x\cdot(-1)}{(y-z)^2} = \frac{x}{(y-z)^2}.$$

Since derivatives compute something related to tangency, you could imagine the partial derivatives compute a similar thing.

Graph of $f(x,y)$



This line, the tangent line of the graph sliced along y-direction, has slope $\frac{\partial f}{\partial y}(a,b)$. In terms of directional vector: $\langle 0, 1, \frac{\partial f}{\partial y}(a,b) \rangle$



This line, the tangent line of the graph sliced along x-direction, has slope $\frac{\partial f}{\partial x}(a,b)$.

In terms of directional vector: $\langle 1, 0, \frac{\partial f}{\partial x}(a,b) \rangle$

Example Find the tangent line of the graph of $f(x,y) = 1 - x^2 - y^2$ to the x-direction at $(1,0,0)$.

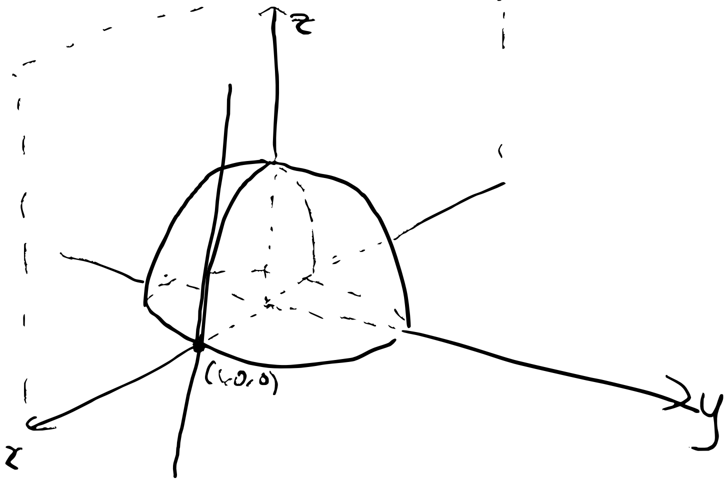
Answer: It is the line that passes through $(1,0,0)$ and has directional vector $\langle 1, 0, \frac{\partial f}{\partial x}(1,0) \rangle$

$$\frac{\partial f}{\partial x} = -2x \Rightarrow \frac{\partial f}{\partial x}(1,0) = -2$$

$\Rightarrow$ directional vector is $\langle 1, 0, -2 \rangle$.

So the line has equation $\langle 1,0,0 \rangle + t\langle 1,0,-2 \rangle$

Picture



$\langle 1,0,0\rangle + \langle 1,0,-2\rangle t$

One can take the implicit differentiation as we do in calculus.

Ex In calculus, if y is implicitly defined as $x^2+y^2=1$, then $\frac{dy}{dx}$ is obtained by taking $\frac{d}{dx}$ on both sides of $x^2+y^2=1$:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$$

which is $2x+2y\frac{dy}{dx} = 0$ (Chain Rule)

$\Rightarrow \quad 2y\frac{dy}{dx} = -2x \qquad \Rightarrow \quad \frac{dy}{dx} = -2x.$

<u>Example</u> If z is implicitly defined as

$x^2-y^2-z^2=1$, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ can be found by taking $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ on both sides of

$x^2-y^2-z^2=1$:

$\frac{\partial}{\partial x}$: $\quad \frac{\partial}{\partial x}(x^2-y^2-z^2) = \frac{\partial}{\partial x}(1)$

$$\Rightarrow \quad 2x + 0 - 2z\frac{\partial z}{\partial x} = 0$$

(y is a constant when doing $\frac{\partial}{\partial x}$)

$$\Rightarrow \quad 2x = 2z\frac{\partial z}{\partial x}$$

$$\Rightarrow \quad \frac{x}{z} = \frac{\partial z}{\partial x}$$

$\frac{\partial}{\partial y}$: $\quad \frac{\partial}{\partial y}(x^2-y^2-z^2) = \frac{\partial}{\partial y}(1)$

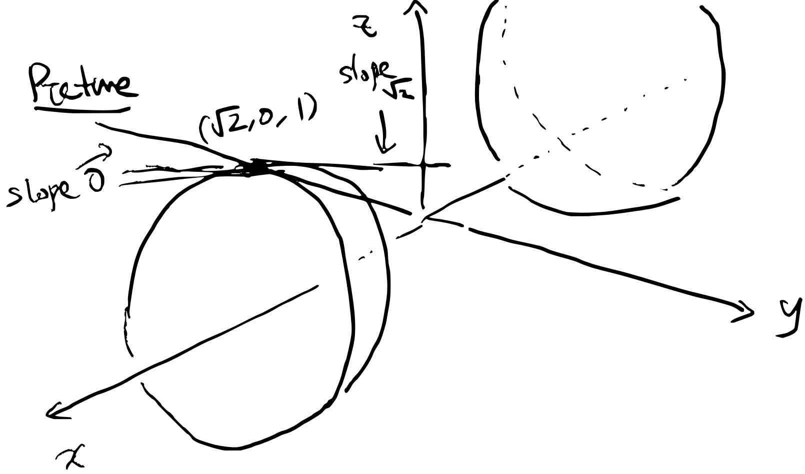
$$\Rightarrow \quad 0 - 2y - 2z\frac{\partial z}{\partial y} = 0$$

(x is a constant when doing $\frac{\partial}{\partial y}$)

$$\Rightarrow \quad -2y = 2z\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad -\frac{y}{z} = \frac{\partial z}{\partial y}.$$

Picture



Higher derivatives

When you have multiple varables, taking the second derivative can have multiple meanings.

$$f(x,y) \to \begin{cases} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} \end{cases}$$

(write in the subscript the order of partial derivatives, $f_{xy} = (f_x)_y$)

Just like usual derivative, one writes $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$, etc.

These are all second partial derivatives.

Example $f(x,y) = x^3 + 2xy^4 - \frac{x}{y}$.

$f_x = 3x^2 + 2y^4 - \frac{1}{y}$ $\qquad$ $f_y = 8xy^3 + \frac{x}{y^2}$

$f_{xx} = 6x$ $\qquad$ $f_{xy} = 8y^3 + \frac{1}{y^2}$ $\qquad$ $f_{yx} = 8y^3 + \frac{1}{y^2}$ $\qquad$ $f_{yy} = 24xy^2 - \frac{2x}{y^3}$

Observe that $f_{xy} = f_{yx}$! This is in general true.

Clairaut's theorem If f is something that you can naturally come up with, $f_{xy} = f_{yx}$. In general, the order of partial derivatives does not matter.

So, $f_{xxy} = f_{xyx} = f_{yxx}$.

Also, this is true for three+ variables.

$f_{xyz} = f_{xzy} = f_{zxy} = \cdots$

__Example__ $f(x,y,z) = xy - yz + xy^2z$

$$f_{xyz} = (f_x)_{yz} = (y + y^2z)_{yz} = (1 + 2yz)_z = 2y$$

$$f_{yzx} = (f_y)_{zx} = (x - z + 2xyz)_{zx} = (-1 + 2xy)_x = 2y$$

$$\vdots$$